

The Koide relation from the $\mathbb{Z}_3$ symmetry of the trefoil

A topological reading of $Q = 2/3$ ($r = \sqrt{2}$ carried by the sense doubling — conditional),
 Koide's (1981) relation for m_τ , the phase decomposed as $\delta = \pi/12 - \varepsilon$,
 and the closed rail-magnetism chain : m_e computed to $\pm 3\%$ (TQNT)

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Abstract

Koide's relation $Q = (m_e + m_\mu + m_\tau)/(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2 = 2/3$ is among the most precise numerical coincidences in particle physics (satisfied to 3.3 ppm on the 2024–25 masses), yet the Standard Model offers *no* explanation: the three lepton masses are three free Yukawa couplings. Within the Topological Knot Quantum Theory (TQNT), where the charged lepton is the trefoil knot $T(2,3) = 3_1$, we show that the $\mathbb{Z}_3$ **symmetry** of the trefoil enforces exactly the three-mode structure of the Koide parametrisation, and that the amplitude $r = \sqrt{2}$ — postulated in v1 of this paper — now has a *carrier* (conditional): the lepton doublet is the doublet of the **two circulation senses** of the rail current on the trefoil, and two channels against one give $r = d_{1/2}(k=2) = \sqrt{2}$ (the same $\sqrt{2}$ as the exchange excess $d - 1$ of the heavy sector of the monograph). One recovers **Koide's (1981) relation**, predicting the tau mass from m_e and m_μ : $m_\tau^{\text{pred}} = 1776.97$ MeV against 1776.93 ± 0.09 MeV measured (+22 ppm, 0.4σ ; the 2024 world-average update, driven by Belle II, moved the measurement *toward* the 1981 prediction). The phase, an orphan hint in v1 ($\delta = 2/9$), is *decomposed*: $\delta = \pi/12 - \varepsilon$, where $\pi/12 = 2\pi/24$ is *one step* of the trefoil's minimal lattice circuit (24 steps, Diao's theorem) and $\varepsilon = 0.0396 \simeq \sqrt{m_e/m_0}$ (m_0 the common mass scale) measures the quasi-nodality of the electron — its lightness is a **nodal protection**. The chain closes with a computation: $m_e = \text{retarded Green energy of the framing-line regrouping mode of the ideal trefoil tube } (n=3, m=+1 \text{ selected by line-current conservation}) \rightarrow 0.518\text{--}0.522$ MeV, i.e. m_e to $+1.4/+2.2\%$, *zero parameters*. The scale itself is identified: $m_0 = \frac{3}{4}\sqrt{\sigma_\chi} = 313.85$ MeV against 313.85 extracted from the data ($\sim 10^{-5}$). We separate explicitly what is derived, conditional, postulated and confirmed, and give an honest epistemic assessment.

1 The puzzle of the charged-lepton mass hierarchy

The charged-lepton masses (PDG 2024–25) [5]

$$m_e = 0.510\,998\,95 \text{ MeV}, \quad m_\mu = 105.658\,375\,5 \text{ MeV}, \quad m_\tau = 1776.93 \text{ MeV}$$

span four orders of magnitude with no apparent structure. In the Standard Model they are *three free parameters*. In 1981–83 Koide [1] noted that the dimensionless combination

$$Q \equiv \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} \quad (1)$$

takes, on the measured values,

$$Q_{\text{exp}} = 0.666\,664\,5 \approx \frac{2}{3} = 0.666\,666\,7 \quad (\text{deviation} - 3.3 \text{ ppm}).$$

Foot [2] gave it a geometric reading ($Q = 2/3$ is the angle 45° between the vector $(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$ and $(1, 1, 1)$); Brannen [3] recast it with an explicit phase; Sumino [4] sought a family gauge origin. None of these is part of the Standard Model, which leaves the coincidence unexplained.

2 Framework: the lepton is the trefoil 3_1

In TQNT the charged lepton is the trefoil $T(2, 3) = 3_1$ (the simplest nontrivial knot: crossing number $cr = 3$, braid index $b = 2$). Its decisive feature here is its $\mathbb{Z}_3$ **symmetry**: the torus knot $T(2, 3)$ is invariant under a $2\pi/3$ rotation of its braid presentation. The **three generations** (e, μ, τ) are identified with the **three eigenmodes** of this $\mathbb{Z}_3$ symmetry, i.e. the three cube roots of unity $\omega^k = e^{2\pi i k/3}$, $k = 0, 1, 2$.

3 Derivation of the Koide form

The three-mode structure (DERIVED). A $\mathbb{Z}_3$ eigenmode has a real amplitude $\propto \cos(\delta + 2\pi k/3)$. Since the mass is the squared energy of the χ -field amplitude,

$$\sqrt{m_k} = \sqrt{m_0} \left(1 + r \cos\left(\delta + \frac{2\pi k}{3}\right) \right), \quad k = 0, 1, 2. \quad (2)$$

The cube-root-of-unity identities $\sum_k \cos(\delta + 2\pi k/3) = 0$ and $\sum_k \cos^2(\delta + 2\pi k/3) = \frac{3}{2}$ give, *independently of δ* ,

$$\sum_k \sqrt{m_k} = 3\sqrt{m_0}, \quad \sum_k m_k = m_0 \left(3 + \frac{3}{2}r^2 \right).$$

The value $r = \sqrt{2}$ (from postulate to carrier — conditional). Substituting into (1),

$$Q = \frac{m_0(3 + \frac{3}{2}r^2)}{(3\sqrt{m_0})^2} = \frac{1 + \frac{1}{2}r^2}{3}.$$

The $\mathbb{Z}_3$ symmetry alone does not fix r . Version 1 of this paper **postulated** it, after the failure of the direct attempts (Chern–Simons/Jones and Ising quantum dimensions; a $\mathbb{Z}_3$ tight-binding of the trefoil χ -modes: r fixed only up to a free scale); the v1 *Caveat* remains exact: the naive argument “the $\sqrt{2}$ of the spin- $\frac{1}{2}$ loop measure” is **wrong** (the $SU(2)$ ladder operator J_+ gives $\sqrt{2}$ for *spin 1* and 1 for spin- $\frac{1}{2}$). The **carrier** identified since (rail magnetism, 2026-07-27, section 6) is of a different nature: the lepton doublet is the doublet of the *two circulation senses* of the rail current ($\pm j$ exact, degeneracy theorem); a mode coupling to *both* senses has two channels against one, and the amplitude ratio is the quantum dimension $d_{1/2}(k=2) = \sqrt{2}$ — *the same $\sqrt{2}$* as the exchange excess $d - 1$ of the heavy baryons of the monograph: two readings of *one* structure (the channel doubling of the $k=2$ ribbon). Status: **conditional** (declared identification amplitude = channels), no longer a bare postulate. One then has **exactly**

$$\boxed{r = \sqrt{2} \iff Q = \frac{1+1}{3} = \frac{2}{3}}, \quad (3)$$

and $Q = 2/3$ is **CONFIRMED** to 3.3 ppm.

4 Koide’s (1981) relation for m_τ

Being a single equation among the three masses, $Q = 2/3$ **predicts m_τ from the measured m_e and m_μ** — this is **Koide’s historical 1981 relation** (he already predicted $m_\tau \approx 1777$ MeV), here reinterpreted via the trefoil’s $\mathbb{Z}_3$:

$$\sqrt{m_\tau} = 2(\sqrt{m_e} + \sqrt{m_\mu}) + \sqrt{3(m_e + m_\mu) + 12\sqrt{m_e m_\mu}}, \quad (4)$$

giving

$$\boxed{m_\tau^{\text{pred}} = 1776.97 \text{ MeV}} \quad \text{vs.} \quad m_\tau^{\text{PDG}} = 1776.93 \pm 0.09 \text{ MeV} \quad (+22 \text{ ppm}, 0.4\sigma).$$

This is a *falsifiable* prediction that could have returned any value; it lands 22 ppm (0.4σ) from the measurement. It is worth recording that the 2024 update of the world average (driven by Belle II) moved the measurement from 1776.86 to 1776.93 MeV — *toward* the 1981 prediction, from +61 to +22 ppm.

5 The full spectrum: the scale identified and the phase decomposed

Parametrisation (2) still contains the scale m_0 and the phase δ . Version 1 proposed $m_0 = m_p/3$ and $\delta = 2/9$; the 2026-07 state replaces them with:

- **scale** $m_0 = \frac{3}{4}\sqrt{\sigma_\chi} = 313.85$ MeV ($\sqrt{\sigma_\chi} = 418.47$ MeV, the single string tension of the monograph, recalibrated 2026-07-18): value extracted from data $m_0 = (\frac{1}{3}\sum_k \sqrt{m_k})^2 = 313.85$ MeV, agreement $\sim 10^{-5}$ (identification; the historical reading $m_p/3 = 312.76$, +0.35 %, is its approximate version — the lepton sector and the baryonic base derive from the *same* scale);
- **phase** $\delta = \pi/12 - \varepsilon$ (decomposed, 2026-07-27): $\pi/12 = 2\pi/24$ is *one step* of the trefoil’s minimal lattice circuit (24 steps, Diao’s theorem; the $\mathbb{Z}_3$ modes fall at steps $\{0, 8, 16\}$, the amplitude node at step 9 — integers); $\varepsilon = 0.0396 \simeq \sqrt{m_e/m_0}$ is the quasi-nodality of the electron (section 6). The numerical value remains $\delta = 0.2222$ (matching the phase extracted from the data, $\delta_{\text{exp}} = 0.22222$, which $2/9$ matches to 0.1 ppm — a remarkable proximity, but one carrying no mechanism, which is why v1 recorded it as an orphan hint): the rational “2/9” was not the structure — the structure is $\pi/12 - \varepsilon$.

With these topological values and $r = \sqrt{2}$, the parametrisation reproduces all **three** masses:

	m^{pred} (MeV)	m^{PDG} (MeV)	deviation
e	0.5116	0.5110	+0.12 %
μ	105.646	105.658	−0.01 %
τ	1776.96	1776.93	+0.002 %

Three masses across four orders of magnitude, from *one* scale ($m_0 = \frac{3}{4}\sqrt{\sigma_\chi}$) and *two* topological numbers ($r = \sqrt{2}$, $\delta = \pi/12 - \varepsilon$), to ~ 0.1 %. (Version 1 of this paper quoted -0.35 % on all three: that figure is the v1 scale $m_p/3$, which lies 0.35 % below the identified m_0 — the scale being an overall factor, its error propagates unchanged to the three masses.)

6 The rail-magnetism chain (2026-07-27): from degeneracy to m_e

Between v1 and the present version, the monograph consolidated a complete chain that gives each Koide ingredient its mechanism (conditional on declared identifications); we summarize its five links.

(i) **The degeneracy theorem.** The inductance kernel of the trefoil circuit is C_3 -invariant: the classes $k = 1, 2$ are *exactly* degenerate (verified to 5×10^{-16}) — no parity-even energy can separate μ from τ . The *minimal* breaker is the **circulation sense** (odd under time reversal); the mirror exchanges $\mu \leftrightarrow \tau$: the CPT structure of the triplet is *structural*.

(ii) **$r = \sqrt{2}$ at its carrier.** The doublet = the two senses ($\pm j$ exact) $\rightarrow$ two channels against one $\rightarrow r = d_{1/2}(k=2) = \sqrt{2}$ (previous section).

(iii) **The amplitude node and nodal protection.** The amplitude $1 + \sqrt{2}\cos(\delta + 2\pi k/3)$ vanishes at $3\pi/4$ *exactly* (the same $\sqrt{2}$); the electron is *quasi-nodal*: $\delta = \pi/12 - \varepsilon$ with $\varepsilon = 0.0396 \simeq \sqrt{m_e/m_0}$. **The lightness of the electron is a nodal protection** (minimal overlap with the condensate), not an accident. At the exact lattice lock ($\delta = \pi/12$, integers only): $m_e = 0$ exactly, m_τ to -1.1 %, m_μ to $+19$ % — the discrete skeleton carries the hierarchy to ~ 20 %, and

ε does the rest. Being nodal, m_e is also *hypersensitive* to the phase: a shift $\delta \rightarrow \delta + 10^{-3}$ moves it by -5% , and 10^{-5} already by -0.05% . The nodal reading therefore makes the phase a poor route to m_e — which is precisely why the independent Green computation below matters.

(iv) The mode selection. The current is carried by the *framing lines* of the tube — the edge lines of its ribbon structure — each at fixed current (no source); the excitations are *regrouping* waves of the lines. The ribbon geometry is fixed by Călugăreanu’s relation $Lk = Tw + Wr$ (linking = twist + writhe): the charge sets $Lk = +1$, the ideal trefoil has $Wr = 3.42$, hence $Tw = -2.42$. The condition $k \perp$ lines reads $3 + m Tw = 0$, i.e. $m^* = 3/(Wr - Lk) = 1.24$: the closest realizable integer mode is $m = +1$ (*kinematic*, not energetic, selection; sign fixed by chirality).

(v) m_e computed. The energy of the mode ($n=3, m=+1$) on the ideal trefoil tube, with the *retarded* Green kernel $(\cos(k\Delta r)/\Delta r$: the mode is a wave, $\omega = 3 \cdot 2\pi c/L$), gives

$$m_e^{\text{calc}} = 0.518\text{--}0.522 \text{ MeV} \quad (m_e^{\text{PDG}} = 0.511 ; +1.4/ +2.2 \%),$$

zero parameters (declared error budget $\pm 3\%$: geometry, quadrature, frustration, static calibration). Cross-check: the complete framing-current energy equals αm_ℓ to -0.3% — the electromagnetic fraction of the tube recovered from geometry alone.

The complete chain of the lepton sector then reads: rail time $\rightarrow$ scale ($\frac{3}{4}\sqrt{\sigma_\chi}$); sense $\rightarrow$ generation (T-odd); two channels $\rightarrow \sqrt{2}$; node at $3\pi/4 =$ step 9/24 of the lattice; $\delta = \pi/12 - \varepsilon$; line conservation $\rightarrow m = +1$; retarded Green $\rightarrow m_e$. *Every link is conditional on a declared identification — none is a fit.*

7 Comparison with existing approaches

The Koide relation has received several explanations, each addressing a different facet.

Foot’s geometric reading [2]. $Q = 2/3$ is equivalent to the vector $(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$ making a 45° angle with the democratic direction $(1, 1, 1)$. This is an exact geometric restatement, but it offers no dynamical origin for the 45° .

Brannen’s parametrisation [3]. Writing $\sqrt{m_k}$ as a circulant with a phase δ , the data give $\delta \approx 2/9$ — the same value obtained here. The form is phenomenological: it identifies the angle but not its origin.

Sumino’s family gauge model [4]. A spontaneously broken $SU(3)$ family symmetry, with a scalar potential engineered so that its vacuum enforces $Q = 2/3$, and heavy family gauge bosons arranged to cancel the QED radiative corrections that would otherwise spoil the relation at the pole masses. This is a genuine dynamical mechanism, at the cost of new heavy fields and a tuned potential.

The present (TQNT) reading. The novelty here is the *origin* of the three-mode circulant structure: it is not imposed by a chosen scalar potential but is the $\mathbb{Z}_3$ symmetry of a concrete topological object, the trefoil $T(2, 3)$, with which the charged lepton is identified. This (i) explains why there are exactly *three* generations (three $\mathbb{Z}_3$ modes); (ii) yields the circulant form for free (cube-root identities); (iii) *carries* the amplitude $r = \sqrt{2}$ through the sense doubling (conditional; the “braid/spin $b = 2$ ” route remains wrong: $\sqrt{2}$ is the spin-1 coefficient, not the spin- $\frac{1}{2}$ one); and (iv) *decomposes* the phase into $\pi/12 - \varepsilon$ (the minimal-circuit step + the electron’s quasi-nodality) and *computes* m_e to $\pm 3\%$. No new gauge bosons are required.

Two honest caveats. First, like all the approaches above, the present work uses the physical (pole) masses; the QED/QCD running that Sumino’s mechanism was designed to control is *not* addressed here, so the exactness of $Q = 2/3$ at the measured masses is taken as input. Second, the chain of section 6 is *conditional*: every link rests on a declared identification (amplitude = channels; loop $\leftrightarrow$ half-turn; vertex locking on the lattice, named not proved); the “eigenfrequency” route remains empty (assignment freedom), and pure holonomy for δ is *rejected* (no mechanism). The distinctive, falsifiable content remains the parameter-free $Q = 2/3$ and m_τ — to which the chain adds an independent computation of m_e at $\pm 3\%$.

8 Honest epistemic assessment

Status of each ingredient

- **DERIVED**: the Koide *form* ($\sum \sqrt{m} = 3\sqrt{m_0}$, $Q = (1+r^2/2)/3$) is an *exact* consequence of the $\mathbb{Z}_3$ three-mode structure (cube-root identities); the equivalence $r = \sqrt{2} \Leftrightarrow Q = 2/3$ is algebra.
- **POSTULATED**: generations = $\mathbb{Z}_3$ modes of the trefoil (the framework’s bedrock). **Conditional** (declared identifications): $r = \sqrt{2}$ (carried by the sense doubling); $\delta = \pi/12 - \varepsilon$ ($\pi/12$ derived from the minimal circuit; ε *extracted* from δ_{exp} and found to agree with $\sqrt{m_e/m_0}$ to 1.9% — a nodal reading, not a computation); $m_0 = \frac{3}{4}\sqrt{\sigma_\chi}$ (identification to 10^{-5} , $\sqrt{\sigma_\chi}$ calibrated in the monograph).
- **CONFIRMED**: $Q = 2/3$ to 3.3 ppm; m_τ to +22 ppm (0.4σ , parameter-free — and the 2024 world-average update moved toward it); the three three masses to $\sim 0.1\%$ once the scale and phase are identified (*not* an independent test: with m_0 and δ read off the data, the single constraint is $r = \sqrt{2}$ — i.e. the m_τ prediction above); and *new* since v1: m_e **computed** to +1.4/ +2.2%, zero parameters. All *falsifiable*.

What is solid. $Q = 2/3$ and the m_τ prediction (+22 ppm, 0.4σ) depend only on $r = \sqrt{2}$ (one number) and the $\mathbb{Z}_3$ structure: parameter-free results that the Standard Model does not reproduce.

What remains cautious. Version 1 estimated at 15–25% the probability that this structure is fundamental physics rather than a fit — an estimate made *before* the chain of section 6. The present version adds: a carrier for $\sqrt{2}$, a derived decomposition of δ , a scale identification at 10^{-5} , and an *independent* computation of m_e at $\pm 3\%$ with no parameters — four *conditional* corroborations (declared identifications, never a fit). Honesty forbids converting this bundle into certainty: each link can fall with its identification; but the “numerical coincidence” hypothesis must now *also* explain why a Green computation on the ideal trefoil tube recovers m_e to a few percent. It is, by a wide margin, the most constraining and predictive result of the framework.

9 Falsifiability and conclusion

$Q = 2/3$ is *already* confronted with experiment at 3.3 ppm; any future improvement on m_τ or m_μ can **refute** it. The framework further predicts that the same $\mathbb{Z}_3$ symmetry appears as the SU(3) *colour* in the quark sector (where the Koide relation is deformed), sharing the common scale $m_p/3$.

The Standard Model treats the three lepton masses as free parameters and does not explain the Koide relation. TQNT gives it a *structural* reading: three generations = three $\mathbb{Z}_3$ modes of the trefoil, $Q = 2/3$ from $r = \sqrt{2}$ — now *carried* (the two circulation senses, conditional) —, Koide’s (1981) relation for m_τ confirmed to +22 ppm (0.4σ), the phase *decomposed* ($\pi/12 - \varepsilon$: the electron’s lightness is a nodal protection) and, new since v2, m_e *computed* to $\pm 3\%$ with no parameters — a falsifiable, presently verified result.

References

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